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ON A PROBLEM OF SPENCER

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Let $X_1, \dots, X_n$ be events in a probability space. Let ϱ_i be the probability X_i occurs. Let ϱ be the probability that none of the X_i occur. Let G be a graph on $[n]$ so that for $1 \leqq i \leqq n$ X_i is independent of $\{X_j | (i,j) \notin G\}$. Let $f(d)$ be the sup of those x such that if $\varrho_1, \dots, \varrho_n \leqq x$ and G has maximum degree $\leqq d$ then $\varrho > 0$. We show $f(1)=1/2$, $f(d)=(d-1)^{d-1}d^{-d}$ for $d \geqq 2$. Hence $\lim_{d\to\infty} df(d)=1/e$. This answers a question posed by Spencer in [2]. We also find a sharp bound for ϱ in terms of the ϱ_i and G.

The probabilistic method is a nonconstructive technique for showing the existence of combinatorial configurations. It works as follows. Suppose we wish to show a class of configurations contains a particular configuration satisfying certain conditions $C_1, C_2, \dots, C_n$. We impose a probability distribution on the class of configurations. We let X_i be the event "condition C_i is violated". We let ϱ_i be the probability that X_i occurs. We let ϱ be the probability that $\overline{X}_1 \cap \overline{X}_2 \cap \dots \cap \overline{X}_n$ occurs (i.e. ϱ is the probability that all of the conditions C_i are satisfied). It is easy to show by induction on n that $\varrho \geqq 1 - \varrho_1 - \varrho_2 - \dots - \varrho_n$. Hence if $\varrho_1 + \varrho_2 + \dots + \varrho_n < 1$ then $\varrho > 0$ which implies the existence of a configuration satisfying all of the conditions $C_1, \dots, C_n$. If all the ϱ_i are equal say $\varrho_1 = \varrho_2 = \dots = \varrho_n = x$ then the condition $\varrho_1 + \varrho_2 + \dots + \varrho_n < 1$ becomes $nx < 1$. It was observed by Lovász [1] that under certain conditions this condition could be weakened and still imply $\varrho > 0$. We say a graph G on $[n]$ is a dependence graph for the events $X_1, \dots, X_n$ if for $1 \leqq i \leqq n$ X_i is independent of $\{X_j | (i,j) \notin G\}$. Suppose G has maximum degree d. Then Lovász showed $4dx \leqq 1$ implies $\varrho > 0$. Spencer [2] noted this could be improved to $e(d+1)x \leqq 1$ implies $\varrho > 0$ or more precisely that $x \leqq d^d/(d+1)^{d+1}$ implies $\varrho > 0$. Spencer defined $f(d)$ as the sup of those x which imply (d fixed) $\varrho > 0$ and asked what is $\lim_{d\to\infty} df(d)$. In this paper we answer this question by finding an exact formula for $f(d)$. First however we give a sharp estimate for ϱ in terms of the ϱ_i and G which is of independent interest.

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Theorem 1. *Let S be a subset of $[n]$. Define*

$$P(S)=\sum_{\substack{S\subseteq T\\ T \text{ ind in } G}}(-1)^{|T|-|S|}\prod_{i\in T}\varrho_i.$$

If $P(S)$ is $\geqq 0$ for all S then $\varrho\geqq P(\emptyset)$. Otherwise $\varrho\geqq 0$. Furthermore these bounds are the best possible. (We interpret the empty sum as 0, the empty product as 1. T ind or independent in G means G contains no edges connecting points in T.)

Proof. Let R be an independent set in G. Note

$$\sum_{R\subseteq S}P(S)=\sum_{R\subseteq S}\sum_{\substack{T\supseteq S\supseteq R\\ T\text{ ind}}}(-1)^{|T|-|S|}\prod_{i\in T}\varrho_i=\sum_{\substack{R\subseteq T\\ T\text{ ind}}}\Big[\prod_{i\in T}\varrho_i\Big](-1)^{|T|}$$

$$\sum_{R\subseteq S\subseteq T}(-1)^{|S|}=\prod_{i\in R}\varrho_i.$$

Letting $R=\emptyset$ this implies $\sum_S P(S)=1$. Hence if $P(S)\geqq 0$ for all S we can let Y be a random variable taking on values S in $\{0,1\}^n$ with probability $P(S)$. Let $Y_1,\dots,Y_n$ be the first through nth components of Y. The probability that $Y_i=1$ is ϱ_i (let $R=\{i\}$). Furthermore it is easy to verify that G is a dependence graph for Y. For $S\subseteq[n]$ let $\alpha(S)$ be the probability that X is 0 on S (i.e. that none of the events $\{X_i|i\in S\}$ occur), $B(S)$ be the probability that Y is 0 on S. Note

$$B(S)=\sum_{S'\subset S}P(S')=$$

$$=\sum_{S'\subseteq S}\sum_{\substack{S'\subset T\\ T\text{ ind in }G}}(-1)^{|T|-|S|}\prod_{i\in T}\varrho_i=\sum_{T\text{ ind}}(-1)^{|T|}\Big(\prod_{i\in T}\varrho_i\Big)\Big(\sum_{S'\subseteq S\cap T}(-1)^{-|S'|}\Big)=$$

$$=\sum_{\substack{T\text{ ind}\\ S\cap T=\emptyset}}(-1)^{|T|}\prod_{i\in T}\varrho_i=\sum_{\substack{T\text{ ind}\\ T\subseteq S}}(-1)^{|T|}\prod_{i\in T}\varrho_i.$$

Now $\varrho=\alpha([n])$ and $P(\emptyset)=B([n])$. Hence we wish to show $\alpha([n])\geqq B([n])$. If $B(S)=0$ for some S then $B([n])=0$ also and $\alpha([n])\geqq B([n])$ trivially. Hence we may assume $B(S)\neq 0$ for all S. Then we claim that if $S_1\subseteq S_2$ then $\dfrac{\alpha(S_1)}{B(S_1)}\leqq\dfrac{\alpha(S_2)}{B(S_2)}$. We prove this by induction on $|S_2|$. First note if $S=\emptyset$, $\alpha(S)=B(S)=1$ and if $S=\{i\}$, $\alpha(S)=B(S)=1-\varrho_i$. It clearly suffices to prove the claim for $|S_2-S_1|=1$. Let $S_2=S_1\cup\{i\}$. Let $S_1=T_1\cup T_2$ where T_1 consists of points of S_1 not adjacent to i in G while T_2 consists of points of S_1 adjacent to i in G. Note $B(S_2)=B(S_1)-\varrho_i B(T_1)$. Furthermore $\alpha(S_2)=$probability (X is 0 on S_2)$=$probability (X is 0 on S_1) — probability (X is 0 on S_1 and 1 on i). Now probability (X is 0 on S_1, 1 on i)$\leqq$probability (X is 0 on T_1, 1 on i)$=\alpha(T_1)\varrho_i$ (by choice T_1, definition dependence graph). Therefore $\alpha(S_2)\geqq\alpha(S_1)-\varrho_i\alpha(T_1)$. Hence

$$\frac{\alpha(S_2)}{B(S_2)}-\frac{\alpha(S_1)}{B(S_1)}\geqq\frac{\alpha(S_1)-\varrho_i\alpha(T_1)}{B(S_1)-\varrho_i B(T_1)}-\frac{\alpha(S_1)}{B(S_1)}=\frac{\varrho_i[B(T_1)\alpha(S_1)-\alpha(T_1)B(S_1)]}{B(S_1)(B(S_1)-\varrho_i B(T_1))}=$$

$$=\frac{\varrho_i B(T_1)}{B(S_1)-\varrho_i B(T_1)}\left[\frac{\alpha(S_1)}{B(S_1)}-\frac{\alpha(T_1)}{B(T_1)}\right]\geqq 0$$

(by induction hypothesis since $T_1 \subseteqq S_1$) as desired. Now $\frac{\varrho}{P(\emptyset)} = \frac{\alpha([n])}{B([n])} \geqq \frac{\alpha(\emptyset)}{B(\emptyset)} = 1 \Rightarrow \varrho \geqq P(\emptyset)$ as desired. Clearly Y shows this bound is the best possible. It remains to show that if $P(S) < 0$ for some S then we can define a random variable $Y = (Y_1, ..., Y_n)$ for which G is a dependence graph, probability $(Y_i = 1)$ is ϱ_i and probability $(Y_1 = Y_2 = ... = Y_n = 0)$ is 0. Let A be a subset of $[n]$ and define (for $S \subseteqq A$).

$$P_A(S) = \sum_{\substack{S \subseteqq T \subseteqq A \\ T \text{ ind in } G}} (-1)^{|T|-|S|} \prod_{i \in T} \varrho_i.$$

Suppose for some A $P_A(\emptyset) < 0$ but $P_A(B) \geqq 0$ for $\emptyset \subset B \subseteqq A$ (as we will see such an A must exist). Let $Q_A(B) = (\prod_{i \in B} \varrho_i)(\prod_{i \in A-B} (1-\varrho_i))$. Choose λ so that $\lambda P_A(\emptyset) + (1-\lambda) Q_A(\emptyset) = 0$. Let Y_A be a random variable taking on values B in $\{0, 1\}^A$ with probability $\lambda P_A(B) + (1-\lambda) Q_A(B)$. Extend Y_A to a random variable Y taking on values $B \cup C$ $(B \subseteqq A, C \subseteqq [n] - A)$ in $\{0, 1\}^{[n]}$ with probability

$$[\lambda P_A(B) + (1-\lambda) Q_A(B)] \prod_{i \in C} \varrho_i \prod_{i \in [n]-A-C} (1-\varrho_i).$$

It is easy to verify that this Y satisfies the given conditions with $\varrho = 0$ as desired. It remains to show such an A exists. Suppose $P_A(B) < 0$ and let C consists of those elements of $A - B$ not adjacent (in G) to any element of B. Then

$$P_A(B) = \sum_{\substack{B \subseteqq T \subseteqq A \\ T \text{ ind in } G}} (-1)^{|T|-|B|} \prod_{i \in T} \varrho_i = \prod_{i \in B} \varrho_i [\sum_{\substack{T \subset C \\ T \text{ ind in } G}} (-1)^{|T|} \prod_{i \in T} \varrho_i] = (\prod_{i \in B} \varrho_i) P_C(\emptyset).$$

This assumes B is independent in G which must be true as otherwise $P_A(B) = 0$ a contradiction. Hence we may choose A so that $P_A(\emptyset) < 0$ but $P_B(\emptyset) \geqq 0$ for any proper subset B of A. Then $P_A(S) \geqq 0$ for any $S \subset A, S \neq \emptyset$. For suppose $P_A(S) < 0$, $S \neq 0$, then by the above $\exists C \subseteqq A - S$ so that $P_C(\emptyset) < 0$ a contradiction. This completes the proof of Theorem 1. ▮

Remark 1. Suppose we take all the ϱ_i to be equal to say q. Then $P(\emptyset)$ is a polynomial of q of degree equal to $\alpha(G)$ (the maximum size of an independent set in G) $\leqq n$. Since finding $\alpha(G)$ in NP-complete in general evaluating $P(\emptyset)$ must also be NP-complete in general. Hence the bound given by the above theorem may be useless in practice.

However there are special cases where we can evaluate the bound in Theorem 1 exactly. Let all the $\varrho_i = x$ and define $F_G(x) = P(\emptyset)$. Let a be a point in G, G_1 be G with a deleted, G_2 be G with a and all its neighbors deleted. Then we have $F_G(x) = F_{G_1}(x) - x F_{G_2}(x)$. Furthermore, if G is the disjoint union of two graphs G_1, G_2, then $F_G(x) = F_{G_1}(x) F_{G_2}(x)$. Now define rooted trees $T_m(n)$ inductively as follows. $T_m(0)$ consists of a single point. $T_m(n+1)$ is formed by connecting the roots of m copies of $T_m(n)$ to the new root. Now applying the above identities (with a=the root) we obtain $F_{T_m(n+1)}(x)=[F_{T_m(n)}(x)]^m-x[F_{T_m(n-1)}(x)]^{m^2}$, $n\geqq 1$. Also $F_{T_m(1)}(x)=$ $=(1-x)^m-x$ and $F_{T_m(0)}(x)=1-x$. Let $a_{-2}=a_{-1}=1$ and $a_{n+1}=a_n^m-xa_{n-1}^{m^2}$. Then $F_{T_m(n+1)}(x)=a_{n+1}$. We ask for what values of x is $a_n>0$ for all n. Let $b_n=$ $=a_n/a_{n-1}^m$ so $b_{-1}=1$, $b_{n+1}=1-x/b_n^m$. It suffices to find those values of x for which $b_n>0$ for all n. Now $b_{n+1}-b_n=x\left(\frac{-1}{b_n^m}+\frac{1}{b_{n-1}^m}\right)$ so $0<b_n\leqq b_{n-1}\Rightarrow b_{n+1}\leqq b_n$. Also $b_0=1-x\leqq 1=b_{-1}$. Hence if the b_n's are all positive they must decrease to a limit say λ. Then $\lambda=1-x/\lambda^m$ or $x=\lambda^m-\lambda^{m-1}$. Now $\lambda^m-\lambda^{m-1}$ is maximized when $\lambda=m/(m+1)$ with value $m^m/(m+1)^{m+1}$. Hence $x\leqq m^m/(m+1)^{m+1}$. Furthermore if $x=\lambda^m-\lambda^{m+1}$ or $\lambda=1-x/\lambda^m$, $m/(m+1)\leqq\lambda\leqq 1$ then $b_n>\lambda\Rightarrow b_{n+1}=1-x/b_n^m>$ $>1-x/\lambda^m=\lambda$ which implies the b_n's decrease to a limit (which must be λ) as $n\to\infty$. Hence we have shown $a_n>0$ for all n iff $x\leqq m^m/(m+1)^{m+1}$. Note the degree of any vertex in $T_m(n)$ is $\leqq m+1$. Hence since Theorem 1 is sharp we have $f(m+1)\leqq$ $\leqq m^m/(m+1)^{m+1}$ $m\geqq 1$ or $f(d)\leqq(d-1)^{d-1}/d^d$, $d\geqq 2$. In fact we have

Theorem 2. $f(d)=(d-1)^{d-1}/d^d$, $d\geqq 2$, $f(1)=1/2$. *Recall $f(d)$ is the* sup *of those x for which maximum degree of G $\leqq d$, $\varrho_1, \varrho_2, \ldots, \varrho_n\leqq x$ imply $\varrho>0$.*

Proof. $f(1)=1/2$ is trivial. We have just seen that letting G be the graphs $T_m(n)$ forces $f(d)\leqq(d-1)^{d-1}/d^d$, $d\geqq 2$. It remains to show that if G is a graph on $[n]$ with each vertex having degree $\leqq d$ $(d\geqq 2)$ and all $\varrho_i\leqq x\leqq(d-1)^{d-1}/d^d$ then $\varrho>0$. Let $\lambda=1-x/\lambda^m$, $m/(m+1)\leqq\lambda\leqq 1$, $m=d-1$. Let $S_2\subseteq[n]$ with $S_2=S_1\cup\{i\}$. Let $S_1=T_1\cup T_2$ where T_1 consists of points of S_1 not adjacent to i in G while T_2 consists of points of S_1 adjacent to i in G. Suppose i has degree $\leqq d-1$ in G restricted to S_2 (i.e. $|T_2|\leqq m$). Define α as in the proof of Theorem 1. Assume $x\geqq 0$. Then we claim $\alpha(S_2)>\lambda\alpha(S_1)$. This follows by induction on $|S_2|$. If $|S_2|=1$ then since $\lambda=1-x/\lambda^m\Rightarrow 1-x>\lambda$ (as $x>0)\Rightarrow\alpha(S_2)=1-\varrho_i\geqq 1-x>\lambda=\lambda\alpha(\emptyset)=\lambda\alpha(S_1)$ as desired. Suppose $|S_2|>1$. As in the proof of Theorem 1 $\alpha(S_2)\geqq\alpha(S_1)-\varrho_i\alpha(T_1)\geqq$ $\geqq\alpha(S_1)-x\alpha(T_1)$. By the induction hypothesis $\alpha(S_1)>\lambda^{|T_2|}\alpha(T_1)$ (since all points of T_2 have degree $\leqq d-1$ in S_1). If $|T_2|=0$ this is not valid but then we have $\alpha(S_2)=$ $=(1-\varrho_i)\alpha(S_1)>\lambda\alpha(S_1)$ as above. Hence $\alpha(S_2)>\left[1-\frac{x}{|T_2|}\right]\alpha(S_1)\geqq(1-x/\lambda^m)\alpha(S_1)=$ $=\lambda\alpha(S_1)$ as desired. Now suppose i has degree d in S_2. Then as above $\alpha(S_2)>$ $>(1-x/\lambda^d)\alpha(S_1)=(2-1/\lambda)\alpha(S_1)$. Now $m\geqq 1$ so $\lambda\geqq 1/2$ so $(2-1/\lambda)\geqq 0$. Hence $\alpha(S_1)>0\Rightarrow\alpha(S_2)>0$. It follows by induction on $|S_2|$ that $\alpha(S_2)>0$. Letting $S_2=[n]$ we have $\varrho=\alpha([n])>0$ as desired. In fact if G is connected we have $\varrho>(2-1/\lambda)\lambda^{n-1}$. (Since if G is connected all induced proper subgraphs of G contain points of degree $\leqq(d-1)$. Hence unless $|S_2|=n$ we can choose i to have degree $\leqq(d-1)$ in G restricted to S_2.) We have assumed $x>0$ but if $x=0$ then clearly $\varrho=1$. This completes the proof of Theorem 2. ∎

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